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THE CONSTRUCTION OF BOUNDED GAME-THEORETIC CONTROLS FOR NON-LINEAR DYNAMICAL SYSTEMS†

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The problem of transferring a non-linear dynamical system, subject to perturbations, to the null equilibrium position in a finite time by means of a bounded control is considered. Only the levels of uncontrollable perturbations are known, and are not assumed to be small. Sufficient conditions are obtained which ensure that the problem has a guaranteed solution for the given domain of initial conditions. An estimate of the guaranteed control time is obtained. The construction of the control can be reduced to the construction of game strategies for auxiliary linear game-theoretic problems. To estimate the "auxiliary noise" in the resulting linear system, the principle of "prescribing and subsequent confirmation" of noise levels is put forward. On the basis of this principle, these estimates are checked on the set of states of the auxiliary linear systems, where the control is also subsequently estimated. As a result, an iterative algorithm for solving the original non-linear problem is obtained. Within the framework of the method proposed a new solution of the game-theoretic problem of the reorientation of an asymmetric rigid body in the presence of noise is given. © 1997 Elsevier Science Ltd. All rights reserved.

This paper develops the approach presented in [1–4] and touches on the study of decomposition [5–7] as well as the partial stabilization and controllability [8–11] of non-linear controlled systems.

1. FORMULATION OF THE PROBLEM

Suppose that the motion of the controlled object can be described by a non-linear system of ordinary differential equations

$$\dot{y}_i = Y_i^{(1)}(\mathbf{x}) + \sum Y_{ik}(\mathbf{x})u_k + Y_i^{(2)}(\mathbf{x})v_i, \quad \dot{z}_j = Z_j(\mathbf{x}) \tag{1.1}$$

Here and below $i = \overline{1, m}$, $j = \overline{1, p}$, $k = \overline{1, r}$ summation over repeated subscripts is performed, $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ is the vector formed by the phase variables y_j, z_j, and $\mathbf{u}$, $\mathbf{v}$ are the vectors formed by the control functions u_k and noise v_i, respectively. The functions $Y_i^{(1)}$, Y_{ik}, $Y_i^{(2)}$, Z_j as well as the derivatives of Z_j with respect to y_j and z_j are defined and continuous in the domain

$$\Lambda: \ |\mathbf{x}| < H = \text{const} > 0$$

Unlike the functions $Y_i^{(1)}$, Z_j that vanish at $\mathbf{x} = \mathbf{0}$, the possibility that Y_{ik} will vanish at $\mathbf{x} = \mathbf{0}$ is excluded (otherwise a contradiction with some conditions to be adopted later on in this paper would arise).

The control functions $\mathbf{u} \in K$ are chosen in the class K of vector-valued functions $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{x}_0)$ ($\mathbf{x}_0$ being the initial value of $\mathbf{x}$) that are piecewise-continuous in $\mathbf{x}$ and $\mathbf{x}_0$ (in the admissible domain of variation of $\mathbf{x}$ and $\mathbf{x}_0$). The control functions $\mathbf{u} \in K$ also satisfy prescribed "geometric" bounds

$$|u_k| \leqslant \alpha_k = \text{const} > 0 \tag{1.2}$$

The noise $\mathbf{v} \in K_1$ can be realized as arbitrary piecewise-continuous functions $\mathbf{v} = \mathbf{v}(t)$ subject to the constraints

$$|v_i(t)| \leqslant \beta_i = \text{const} > 0 \tag{1.3}$$

Problem 1. It is required to find control functions $\mathbf{u} \in K$ that transfer system (1.1) from the given

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domain S of initial perturbations $\mathbf{x}_0 = \mathbf{x}(t_0) \in S$ to the equilibrium position $\mathbf{x}_1 = \mathbf{x}(t_1) = \mathbf{0}$ in a *finite* time for any $\mathbf{v} \in K_1$. The time $t_1 > t_0$ is not fixed.

2. AUXILIARY LINEAR SYSTEM

We consider the vector-valued function $\Phi(\mathbf{x}, \mathbf{u})$: $R^{n+r} \to R^{p+q}$, $n = m + p$ (q is a constant and $0 \leqslant q < r \leqslant m$) with components (here and henceforth $s = \overline{1, q}$)

$$\Phi_s = \sum Y_{sk}(\mathbf{x})u_k, \quad \Phi_{q+j} = \sum[\partial Z_j(\mathbf{x}) / \partial y_i][\sum Y_{ik}(\mathbf{x})u_k] \tag{2.1}$$

Assuming that $p + q \geqslant r$, we introduce the Jacobi matrix $F = \| \partial\Phi/\partial\mathbf{u} \|$. Suppose that in the domain

$$\Lambda_1: \ |\mathbf{x}| < H_1 = \text{const} > 0 \quad (\Lambda_1 \subseteq \Lambda) \tag{2.2}$$

the condition

$$\operatorname{rank} F(\mathbf{x}) = r \tag{2.3}$$

is satisfied. Moreover, we assume that the same rows of F, say, the first r rows of F, are linearly independent for all $\mathbf{x} \in \Lambda_1$.

In the case (2.3) the system $\Phi_k(\mathbf{x}, \mathbf{u}) = u_k^*$ (u_k^* being auxiliary control functions defined below, which form a vector $\mathbf{u}^*$) has a solution

$$\mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}^*) \tag{2.4}$$

with $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, the components f_k of the vector-valued function $\mathbf{f}$: $R^{n+r} \to R^r$ being continuous with respect to $\mathbf{x}$ and $\mathbf{u}^*$ for $\mathbf{x} \in \Lambda_1$ and all $\mathbf{u}^*$.

We introduce the notation

$$v_s^* = \hat{Y}_s(\mathbf{x}, \mathbf{v}), \quad v_{q+l}^* = \langle [\partial Z_l(\mathbf{x}) / \partial \mathbf{x}] \hat{\mathbf{X}}(\mathbf{x}, \mathbf{v}) \rangle \tag{2.5}$$

Here and henceforth $l = \overline{1, r-q}$. The $\hat{Y}_i$ can be obtained from the expressions for $\dot{y}_i$ in (1.1) for $u_k = 0$. The vector-valued function $\hat{\mathbf{X}}$: $R^{n+m} \to R^n$ has components $\hat{Y}_l$, Z_j and $\partial Z_l/\partial \mathbf{x}$: $R^n \to R^n$ has components $\partial Z_l/\partial y_i$, $\partial Z_l/\partial z_j$. The scalar product is denoted by $\langle \cdot \rangle$.

We shall treat v_k^* as "auxiliary noise". As a result, under the above assumptions one can extract the following linear conflict-controlled system from the closed non-linear system (1.1), (2.4)

$$\dot{y}_s = u_s^* + v_s^* \tag{2.6}$$

$$\ddot{z}_l = u_{q+l}^* + v_{q+l}^* \tag{2.7}$$

where u_k^* is the control.

On the basis of the solution of the corresponding game-theoretic problems for the auxiliary linear system (2.6), (2.7), below we shall construct a solution of the original non-linear Problem 1. As a result, (2.4) can be regarded as the general structural form of control in Problem 1. The parameters of this form, i.e. the auxiliary control u_k^* can be determined by solving the corresponding linear game-theoretic problems.

Unlike the formulations of problems considered before [3, 4], the structure of the "auxiliary noise" v_k^* becomes more complicated as one changes from the original system (1.1) to the auxiliary linear system (2.6), (2.7). This enables us to simplify the control u_k obtained before [3, 4], but it becomes more difficult to find v_k^*. As will be shown later, in specified problems the proposed principle of "prescribing and subsequent confirmation" of v_k^* may be useful in this respect.

3. AUXILIARY LINEAR GAME-THEORETIC PROBLEMS

For system (2.6), (2.7) we shall solve the problem of the fastest transfer to the position

$$y_s = 0, \quad z_l = \dot{z}_l = 0 \tag{3.1}$$

by means of the auxiliary controls u_k^* for any admissible v_k^*.

We shall consider this problem as a differential game. One of the players, who has u_k^* at his disposition, tries to minimize the time τ taken to transfer to (3.1). The other player (the "opponent"), who tries to maximize τ, can control the "auxiliary noise" v_k^*.

For the problem to be solvable the admissible levels of u_k^* must exceed v_k^*. Following (1.2) and (1.3) we write the appropriate restrictions as

$$|u_k^*| \leqslant \alpha_k^* = \text{const} > 0, \quad |v_k^*| \leqslant \beta_k^* = \rho_k \alpha_k^*, \quad 0 < \rho_k < 1 \tag{3.2}$$

A procedure for determining the levels α_k^*, β_k^* is considered below. Here we consider them as given, so that (3.2) is satisfied.

The solution of the differential game for system (2.6), (2.7) subject to restriction (3.2) can be reduced [5, 12] to the problem of the optimal speed of response for the system

$$\dot{y}_s = (1 - \rho_s) u_s^* \tag{3.3}$$

$$\ddot{z}_l = (1 - \rho_{q+l}) u_{q+l}^* \tag{3.4}$$

subject to restrictions $|u_k^*| \leqslant \alpha_k^*$.

The boundary conditions are the same as for system (2.6), (2.7). System (3.3), (3.4) can be obtained from (2.6), (2.7) for $v_k^* = -\rho_k u_k^*$. (This is the "worst" v_k^*-optimal control for the "opponent".)

The solution of the time-optimization problem for systems of type (3.3), (3.4) has the form [13]

$$u_s^* = \begin{cases} -\alpha_s^* \operatorname{sign} y_s, & y_s \neq 0 \\ 0, & y_s = 0 \end{cases} \tag{3.5}$$

$$u_{q+l}^* = \begin{cases} \alpha_{q+l}^* \operatorname{sign} \psi_l(z_l, \dot{z}_l), & \psi_l \neq 0 \\ \alpha_{q+l}^* \operatorname{sign} z_l = -\alpha_{q+l}^* \operatorname{sign} \dot{z}_l, & \psi_l = 0 \end{cases} \tag{3.6}$$

Here $\psi_l = -\dot{z}_l - [2\alpha_{q+l}^*(1 - \rho_{q+l})]^{-1} z_l |z_l|$ are switching functions.

The number $\tau = \max(\tau_k)$, where τ_k is the optimal time for each subsystem of system (3.3), (3.4), defines the minimum guaranteed time in the game-theoretic problem for system (2.6), (2.7) under consideration. If v_k^* differ from the "worst" control, the time taken to transfer to the state (3.1) does not exceed τ.

We will consider some features of the phase trajectories of system (2.6), (2.7), (3.5), (3.6) for $v_k^* \not\equiv -\rho_k u_k^*$. In the case of (2.6), (3.5) for all admissible v_s^* for system moves along a curve (a straight line if $v_s^* \equiv 0$) which lies between rectilinear segments, namely, the trajectories of (2.6), (3.5) for $v_s^* = \pm\rho_s u_s^*$, respectively, until the point $y_s = 0$ is reached.

In the case of (2.7), (3.6) the system initially moves (until the switching curve $\psi_l = 0$ is reached) between the parabolic arches of the systems $\ddot{z}_l = (1 \pm \rho_{q+l}) u_{q+l}^*$ for u_{q+l}^* of the form (3.6). Then, once it reaches the switching curve, it slides along it until the required value is attained.

4. AN APPROACH TO THE SOLUTION OF PROBLEM 1

We denote by $\mathbf{X}_\mathbf{v} = \{\mathbf{x}_\mathbf{v}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{v})\}$ the set of processes of system (1.1), (2.4), (3.5), (3.6) corresponding to all $\mathbf{v} \in K_1$. Extracting the linear system (2.6), (2.7) from (1.1), (2.4) we can find the sets $\mathbf{Y}_\mathbf{v}^* = \{y_s(t; t_0, \mathbf{x}_0, \mathbf{v})\}$ and $\mathbf{Z}_\mathbf{v}^* = \{z_l(t; t_0, \mathbf{x}_0, \mathbf{v})\}$ for some of the components (y_s, z_l) of $\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{v}) \in \mathbf{X}_\mathbf{v}$ as the sets of processes of the linear systems (2.6), (3.5) and (2.7), (3.6) corresponding to all admissible v_k^*.

Denoting by $\mathbf{y}^*$, $\mathbf{z}^*$ the vectors formed by y_s and z_l, respectively, we conclude that when solving Problem 1 we first solve the problem of transferring system (1.1) to $\mathbf{x} = \mathbf{0}$ with respect to a part of the variables, namely, $\mathbf{y}^*$, $\mathbf{z}^*$. The control problem with respect to a part of the variables can be reduced to the corresponding control problem with respect to all the variables for the auxiliary linear system (3.3), (3.4). As a result, as in stability analysis with respect to a part of the variables, in the case of control with respect to $\mathbf{y}^*$, $\mathbf{z}^*$, the variables forming the vector $\dot{\mathbf{z}}^*$ are to be included in the set of controlled variables [14].

We will impose auxiliary conditions under which the realized transition of system (1.1) to $\mathbf{x} = \mathbf{0}$ with respect to a part of the variables is in fact a solution of the original Problem 1 of control with respect to all the variables. To do this we assume that $r-q \geqslant m+p-r$. Along with F we also introduce the Jacobi matrix Q of Z_l with respect to the variables forming the vectors $\mathbf{y}_*$ and $\mathbf{z}_*$ (these include y_i and z_l apart from y_s and z_l, respectively). We put $\mathbf{x}^* = (\mathbf{y}^*, \mathbf{z}^*)$, $\mathbf{x}_* = (\mathbf{y}_*, \mathbf{z}_*)$, so that $\mathbf{x} = (\mathbf{x}^*, \mathbf{x}_*)$ as a result.

Suppose that in the domain

$$\Lambda_2\colon\ |\mathbf{x}| < H_2 = \text{const} > 0 \quad (\Lambda_2 \subseteq \Lambda) \tag{4.1}$$

the condition

$$\operatorname{rank} Q(x) = m + p - r \tag{4.2}$$

holds. Furthermore, assume that the same rows of Q are independent for all $\mathbf{x} \in \Lambda_2$.

We observe that for $r \leqslant m$ the inequalities $p+q \geqslant r$ and $r-q \geqslant m+p-r$ are satisfied simultaneously if

$$p + q = r = m \tag{4.3}$$

In this case $m+p-r = r-q$ and Q is a square matrix, so that the condition of linear independence of the same rows of this matrix for all $\mathbf{x} \in \Lambda_2$, which was imposed in addition to (4.2), is unnecessary.

Condition (4.3) holds, for example, when (1.1) is used as a model of the angular motion of a rigid body. Moreover, this condition can be relaxed within the framework of the method proposed.

When (4.2) is satisfied, the equalities $\dot z_l = Z_l(\mathbf{x})$ in Λ_2 admit of the solution (Ψ: $R^{2r-q} \to R^{n-r}$ is a continuous function in Λ_2)

$$\mathbf{x}_* = \Psi(\mathbf{x}^*,\ \dot{\mathbf{z}}^*),\quad \Psi(\mathbf{0},\ \mathbf{0}) = \mathbf{0} \tag{4.4}$$

Equalities (4.4) relate the components of $\mathbf{X}_\mathbf{v}$ that do not appear in $\mathbf{Y}^*_\mathbf{v}$ and $\mathbf{Z}^*_\mathbf{v}$ to those of $\dot z_l = Z_l(\mathbf{x})$. The phase vector $\mathbf{x}$ of (1.1) in Λ_2 can be represented as

$$\mathbf{x} = [\mathbf{y}^*, \mathbf{y}_*(\boldsymbol{\mu}), \mathbf{z}^*, \mathbf{z}_*(\boldsymbol{\mu})] \triangleq \mathbf{W}(\boldsymbol{\mu}), \quad \boldsymbol{\mu} = (\mathbf{y}^*, \mathbf{z}^*, \dot{\mathbf{z}}^*)$$

It follows that

$$\mathbf{X}_\mathbf{v} = \{\mathbf{W} = \mathbf{W}[t; t_0, \mathbf{W}_0(\mathbf{y}_0^*, \mathbf{z}_0^*, \dot{\mathbf{z}}_0^*(\mathbf{x}_0)), \mathbf{v}]\} \triangleq \{\mathbf{W}_\mathbf{v}(t) = \mathbf{W}(t; t_0, \mathbf{x}_0, \mathbf{v})\}$$

Hence $\mathbf{X}_\mathbf{v}$ can be estimated by means of bound for the sets of the processes $\mathbf{y}^*(t; t_0, \mathbf{y}_0^*, \mathbf{v})$, $\mathbf{z}^*[t; t_0, \mathbf{z}_0^*, \dot{\mathbf{z}}_0^*(\mathbf{x}_0), \mathbf{v}]$, $\dot{\mathbf{z}}^*[t; t_0, \mathbf{z}_0^*, \dot{\mathbf{z}}_0^*(\mathbf{x}_0), \mathbf{v}]$ of the linear (rather than the original non-linear) system (2.6), (2.7), (3.5), (3.6), respectively. As a result, when (4.4) is satisfied, the transfer of (1.1) to the position $\mathbf{x} = \mathbf{0}$ with respect to $\mathbf{y}^*$ and $\mathbf{z}^*$ is indeed a solution of the original Problem 1 of control with respect to all the variables (with respect to $\mathbf{x}$).

5. BASIC RESULTS

Let us summarize the above discussion.

Theorem 1. Suppose that the following conditions are satisfied: (1) there is a constant q ($0 \leqslant q < r$) such that $p+q = r = m$; (2) condition (2.3) in the domain (2.2); (3) condition (4.2) in the domain (4.1); (4) for the "assigned" values τ, β_k^* (and the values α_k^* predetermined by them) the estimates

$$|\mathbf{W}_\mathbf{v}(t)| < \min(H_1, H_2) \tag{5.1}$$

$$|\upsilon_k^*[\mathbf{W}_\mathbf{v}(t), \mathbf{v}]| \leqslant \beta_k^*,\quad |f_k[\mathbf{W}_\mathbf{v}(t), \mathbf{u}^*]| \leqslant \alpha_k$$

hold for any $\mathbf{x}_0 \in S$.

Then for all $\mathbf{x}_0 \in S$, relations (2.4), (3.5) and (3.6) solve Problem 1. Moreover, it is guaranteed that system (1.1) will be transferred precisely to the position $\mathbf{x} = \mathbf{0}$ in a finite time τ by means of $\mathbf{u} \in K$ for any $\mathbf{v} \in K_1$.

Proof. If condition (2.3) is satisfied in Λ_1, system (2.6), (2.7) can be obtained from the original non-linear system (1.1), (2.4) by an appropriate non-linear transformation of variables. Under this transformation the state of the variables y_s and z_l of system (1.1), (2.4) in Λ_1 will be completely defined by the state of the same variables of system (2.6), (2.7). Thus, for all $\mathbf{x}_0 \in S_1 = \{\mathbf{x}_0: |\mathbf{x}_v(t)| < H_1\}$ the control (2.4), (3.5), (3.6) guarantees that system (1.1) will be transferred precisely to the position

$$y_s = 0, \quad z_l = Z_l(\mathbf{x}) = 0 \tag{5.2}$$

in a finite time τ.

When condition (4.2) is satisfied in Λ_1, (4.4) holds. Taking $\dot{z}_l = Z_l(\mathbf{x})$ into account, by the continuity of the vector-valued function $\mathbf{\Psi}$ and the condition $\mathbf{\Psi}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ the modulus of this function can be made as small as desired by reducing the moduli y_s, z_l and $Z_l(\mathbf{x})$. Thus, for all $\mathbf{x}_0 \in S_2 = \{\mathbf{x}_0: |\mathbf{x}_v(t)| < \min(H_1, H_2)\}$ the control (2.4) guarantees that system (1.1) will be transferred not only to (5.2) but also precisely to the position $\mathbf{x} = \mathbf{0}$.

As a result of (1.2), the admissible set of initial perturbations $\mathbf{x}_0$ must also satisfy the condition $\mathbf{x}_0 \in S_3 = \{\mathbf{x}_0: |f_k[\mathbf{x}_v(t), \mathbf{u}^*]| \leqslant \alpha_k\}$. Besides, for the "assigned" levels β_k^* to be confirmed it is necessary that $\mathbf{x}_0 \in S_4 = \{\mathbf{x}_0: |v_k^*[\mathbf{x}_v(t), \mathbf{v}]| \leqslant \beta_k^*\}$.

When conditions 1–4 of the theorem are satisfied, the given domain S of initial perturbations will be contained in the intersection of the sets $S_1, \ldots, S_4$. Given that a function $\mathbf{W} = \mathbf{W}(\mathbf{y}^*, \mathbf{z}^*, \dot{\mathbf{z}}^*)$: $R^{2r-q} \to R^n$ is introduced, this means that for all $\mathbf{x}_0 \in S$ the control (2.4), (3.5), (3.6) guarantees that system (1.1) can be transferred precisely to the position $\mathbf{x} = \mathbf{0}$ in a finite time τ for all $\mathbf{v} \in K_1$. Then the control (2.4), (3.5), (3.6) satisfies the given condition (1.2). The theorem has been proved.

Let us indicate how to relax the assumptions of Theorem 1.

Suppose that N first integrals $R_\varepsilon(\mathbf{x}) = \text{const}$, $R_\varepsilon(\mathbf{0}) = 0$ of (1.1) are known. Assuming that $r - q + N \geqslant m + p - r$, in place of Q we introduce the Jacobi matrix Q^* of Z_l and R_ε with respect to the variables in $\mathbf{y}_*$ and $\mathbf{z}_*$.

Corollary 1. Suppose that (1) there is q $(0 \leqslant q < r)$ such that $r \leqslant p + q \leqslant 2r - m + N$; (2) rank $Q^* = m + p - r$ for $\mathbf{x} \in \Lambda_2$. If conditions 2 and 4 of Theorem 1 are satisfied, then (2.4), (3.5) and (3.6) solve Problem 1 for all $\mathbf{x}_0 \in S$.

Another way of relaxing the conditions of Theorem 1 is to use several constructions of equations of the type (2.4) successively. Then system (1.1) will transfer in stages (the total time being finite) from $\mathbf{x}_0 \in S$ to $\mathbf{x} = \mathbf{0}$.

6. AN ALGORITHM FOR SOLVING PROBLEM 1

The above approach to solving Problem 1 involves the following stages.

1. The choice of construction (2.4) of the controls u_k, where u_k^* have the form (3.5), (3.6). At this stage α_k^*, β_k^* and, consequently, the parameters α_k^*, ρ_{q+l} in (3.5), (3.6) are not bounded. The structure of u_k is simplified compared to the earlier approach presented in [1–4]

2. Preliminary selection of the guaranteed control time τ and "prescribing" of the levels β_k^*. This predetermines α_k^*, ρ_q and specifies the parameters in (3.5), (3.6). When $\tau_k = \tau$ ("equalization" of the control time for each variable y_s and z_l of system (3.3), (3.4)) the numbers α_k^* can be determined from the expressions (see [5])

$$\tau = [\alpha_s^*(1-\rho_s)]^{-1}|y_{s0}|, \tag{6.1}$$

$$\tau = [\alpha_{q+l}^*(1-\rho_{q+l})]^{-1}\{[\tfrac{1}{2}Z_{l0}^2 - \alpha_{q+l}^*(1-\rho_{q+l})z_{l0}\operatorname{sign}\psi_l]^{1/2} - Z_{l0}\operatorname{sign}\psi_l\}$$

3. Estimation of the phase variables of system (2.6), (2.7), (3.5), (3.6) on the set L of states of this system for all admissible $|v_k^*| \leqslant \beta_k^*$. Verifying that the inequalities $|v_k^*| \leqslant \beta_k^*$ are indeed satisfied on L. (A knowledge of the levels β_i of the "original" noise v_i is used.) Thus, at this stage we propose the principle of "prescribing and subsequent verification" of the "auxiliary noise" levels β_k^*.

4. Verifying (5.1) on L for $\mathbf{x}_0 \in S$. We observe that for the structure of (2.4) it is natural to expect that max $|u_k|$ will be close on the set L and on the subset $L^* \subset L$ of the states of the linear system (2.6), (2.7), (3.5), (3.6) for various combinations of $v_k^* = \pm\rho_k u_k^*$ and $v_k^* \equiv 0$.

If the inequalities $|v_k^*| \leqslant \beta_k^*$ or (1.2) are not satisfied on L or, conversely, they are satisfied with a "margin", then the search for proper values of τ and β_k^* is continued. Otherwise the guaranteed control time is governed by the choice of τ.

As a result, we obtain an iterative algorithm for solving Problem 1. The solvability of this algorithm depends on the relationship between the levels of the "original" control functions u_k and the noise v_i as well as on the possible values of $\mathbf{x}_0$ and τ. The corresponding solvability conditions are given in Section 5.

Note that the controls $\mathbf{u} \in K$ preserve the dependence on the initial state $\mathbf{x}_0$ also, even though they are constructed as functions of the actual phase vector $\mathbf{x}$. This is due to the constants α_k^* and ρ_{q+l} in the expressions for $\mathbf{u}$, which depend on $\mathbf{x}_0$.

7. "INVERSE" VERSION OF THE ALGORITHM

1. Choose τ and β_k^*.
2. Compute α_k^* (using (6.1)) and verify that the values v_k^* on L satisfy $|v_k^*| \leq \beta_k^*$.
3. Estimate the levels of the controls u_k using (2.4). This enables the capabilities of the controls u_k of the form (2.4) to be estimated.

8. APPLICATION TO THE GAME-THEORETIC PROBLEM OF THE REORIENTATION OF AN ASYMMETRIC RIGID BODY

We consider the dynamical Euler equations (one equation is written down, from which the other ones can be obtained by cyclic permutation of the subscripts $1 \to 2 \to 3$)

$$A_1\dot{y}_1 = (A_2 - A_3)y_2y_3 + u_1 + v_1 \quad (1\ 2\ 3) \tag{8.1}$$

governing the angular motion of a rigid body about the centre of mass. Here y_i are the projections of the angular velocity vector of the rigid body onto its principal central axes of inertia, u_k are the projections of the control moment onto these axes, and A_i are the principal central moments of inertia. The moments v_i characterize the external forces and uncontrollable perturbations. Even though $m = r = 3$ in the case in question, the two subscripts i and k will be preserved, so that (8.1) corresponds to the first group of Eqs (1.1) and the relationships obtained earlier can be used.

Along with (8.1) we consider the kinematic equations governing the body orientation in terms of the Rodrig–Hamilton variables [15]

$$2\dot{z}_1 = y_1z_4 + y_3z_2 - y_2z_3 \quad (1\ 2\ 3) \tag{8.2}$$

The variable z_4 in (8.2) is related to z_j ($j = 1, 2, 3$) by

$$z_4^2 + \Sigma z_j^2 = 1 \tag{8.3}$$

and an equation for $\dot{z}_4$ can be obtained if required.

For system (8.1)–(8.3) $m = p = r = 3$ and (4.3) holds for $q = 0$. Henceforth $\mathbf{z} = (z_4, z_1, z_2, z_3)$ and $\mathbf{x} = (\mathbf{y}, \mathbf{z})$.

8.1. *Formulation of the reorientation problem.* We choose the control functions $\mathbf{u} \in K$ in the class K of piecewise continuous functions $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{x}_0)$ with bounds (1.2). In the case in question inequalities (1.2) correspond to three pairs of engines fixed to the body.

The noise $\mathbf{v} \in K_1$ can be realized as arbitrary piecewise continuous functions $\mathbf{v} = \mathbf{v}(t)$ subject to (1.3).

Problem 2. It is required to find controls $\mathbf{u} \in K$ that transfer the body from an arbitrary initial state $\mathbf{z}(t_0) = \mathbf{z}_0$ to a prescribed state $\mathbf{z}(t_1) = \mathbf{z}_1$ in finite time for any $\mathbf{v} \in K_1$. Both states are stationary $\mathbf{y}(t_0) = \mathbf{y}_0 = \mathbf{y}(t_1) = \mathbf{y}_1 = \mathbf{0}$. The time t_1 is not fixed and is to be found from speed of response requirements.

Without loss of generality we can assume that $\mathbf{z}_1 = (1, 0, 0, 0)$. Indeed, in this case orientation can be regulated relative to a reference system specified at the initial instant of time. Everywhere below $i, j, k = 1, 2, 3$ in accordance with the adopted notation.

A game-theoretic approach to the problem of the reorientation of an asymmetric rigid body was proposed in [2]. This problem is topical, for example, in the dynamics of spacecraft, robotics and biomechanics. It is necessary to obtain guaranteed solutions that are traditional in game theory because of the increasing practical requirements. However, a rigorous solution of non-linear game-theoretic problems is very difficult to obtain even if modern computers are used. In this connection a method was also proposed in [2] which enables the given non-linear problem to be reduced to linear game-theoretic problems. It is based on the choice of the structure of controls, of the type presented in [16–18], which was independently proposed in [19] from a slightly different viewpoint. The resulting control is robust, ensuring exact reorientation of the body in a finite time, which can be computed from the time-optimization condition.

The control functions in [2] are non-linear functions of variables defining both the angular velocity and the orientation of the body. For any admissible realizations of noise (except for special cases) they represent five impulses of variable intensity.

Following the proposed modification of the approach presented in [1–4], the method described in [2] is modified below to obtain simpler controls, which are consequently easier to realize. In particular, it does not contain components compensating for the gyroscopic moments of the body $M_1 = (A_2 - A_3)y_2y_3$ (1 2 3). As in [2], the solution can be reduced to simpler linear game-theoretic problems. The control is simplified on account of a more complex "auxiliary noise" structure in the resulting linear systems.

Estimates for the "auxiliary noise" require a computation on the set of states of linear auxiliary conflict-controlled systems. Unlike the method described in [2], to obtain such estimates the proposed principle of "prescribing and subsequent confirmation" of noise levels must be used.

Despite this difference, the proposed modification of the method described in [2] does not involve any more complex calculations. Besides, for certain restrictions on the control functions the computations are much simplified.

As regards the game-theoretic problem of the controlled angular motion of a rigid body under consideration, the papers [20–26] on the controlled motion of an aircraft in a medium with uncertain parameters should be mentioned.

8.2. *Auxiliary conflict-controlled system*. We differentiate both sides of each of the equations for $\dot{z}_j$ in (8.2) with respect to time and replace $\dot{y}_j$ by the expressions from (8.1). After some reduction we obtain

$$\ddot{z}_1 = f_1(\mathbf{z}, \mathbf{u}) + \varphi_1(\mathbf{y}, \mathbf{z}, \mathbf{v}) \tag{8.4}$$

$$f_1 = \tfrac{1}{2}(z_4 u_1 A_1^{-1} + z_2 u_3 A_3^{-1} - z_3 u_2 A_2^{-1})$$

$$\varphi_1 = \tfrac{1}{2}[z_4(v_1 + M_1)A_1^{-1} + z_2(v_3 + M_3)A_3^{-1} - z_3(v_2 + M_2)A_2^{-1}] - \tfrac{1}{4} z_1 \Sigma y_i^2 \quad (1\ 2\ 3)$$

We will regard f_j and φ_j, respectively, as auxiliary control functions u_j^* and noise v_j^*. As a result, expressions (8.4) can be regarded as a conflict-controlled system of type (2.7)

$$\ddot{z}_j = u_j^* + v_j^* \tag{8.5}$$

Then the "original" controls u_k can be expressed in terms of u_j^* by equalities of type (2.4)

$$u_1 = \frac{2A_1}{z_4}[(z_4^2 + z_1^2)u_1^* + (z_1 z_2 + z_4 z_3)u_2^* + (z_1 z_3 - z_4 z_2)u_3^*] \quad (1\ 2\ 3) \tag{8.6}$$

Construction (8.6) can be regarded as the general structural form of the controls in Problem 2. The parameters of this form, that is, the auxiliary controls u_k^* can be determined when solving the corresponding linear game-theoretic problems.

Structure (8.6) of the controls involves the factor z_4^{-1}, which formally leads to a "singularity". However, the following more detailed analysis shows that in the case when $\mathbf{z}_1 = (1, 0, 0, 0)$ the relation $z_4 \in [z_{40}, 1]$ holds during the control process. Thus, the "singularity" simply does not appear. If z_{40} is small or $z_1 \neq (1, 0, 0, 0)$, it suffices to change to the control functions (or a combination thereof) obtained from (8.6) by a transposition of subscripts. The "suboptimal" control functions u_k are finally chosen inductively, which is characteristic of many modern methods of applied control theory. We emphasize that the iterative search for "suboptimal" controls described is quite simple within the framework of the proposed algorithm for solving Problem 2. It can be realized in real time while the controlled object is functioning.

8.3. *Auxiliary game-theoretic control problems*. For system (8.5) we will solve the problem of transferring it to the position

$$z_j = \dot{z}_j = 0 \tag{8.7}$$

in the minimum time by means of auxiliary controls u_j^* for any admissible noise v_j^*.

This problem can be treated as a time-minimax differential game of the type considered in Section 3. Its solution, subject to restrictions of the type (3.2), can be reduced to the time-optimization problem (wit the same boundary conditions) for a system of type (3.4).

As before, the procedure of selecting α_j^*, β_j^* involves the principle of "prescribing and subsequent confirmation" of these levels. For the time being, we assume that these are given, so that conditions of type (3.2) are satisfied.

Since $z_{j0} = \dot{z}_{j0} = 0$ (which follows from the fact that $\mathbf{y}_0 = \mathbf{y}_1 = \mathbf{0}$), the quantity

$$\tau = \max(\tau_j), \quad \tau_j = 2\{|z_{j0}|[\alpha_j^*(1-\rho_j)]^{-1}\}^{1/2} \tag{8.8}$$

defines the minimum guaranteed time τ taken to reach the position (8.7). If $v_j^* \not\equiv -\rho_j u_j^*$, the time taken to reach the position (8.7) does not exceed τ.

Henceforth we shall assume that the subscripts in (3.6) correspond to those in (8.5), i.e. $q + l$ is replaced by j in (3.6). An analysis of the phase portrait of the system (8.5), (3.6) shows that the condition $z_4 > \gamma = \text{const} > 0$, being satisfied for $t = t_0$, holds for any admissible v_j^* for $t \in [t_0, t_0 + \tau]$. This is important from the point of view of the operating efficiency of the controls u_k of the form (8.6).

8.4. *Algorithm for solving Problem* 2. We first solve the problem of transferring the body to the required position with respect to z_j. This problem can be reduced to the game-theoretic control problem (with respect to all the variables) for the linear system (8.5). Solving the system for $\dot{z}_j$ in (8.2) as an algebraic system with respect to y_i, we obtain equalities of the type (4.4)

$$y_1 = \frac{2}{z_4}[(z_4^2 + z_1^2)\dot{z}_1 + (z_1z_2 + z_4z_3)\dot{z}_2 + (z_1z_3 - z_4z_2)\dot{z}_3] \quad (1\ 2\ 3) \tag{8.9}$$

On the basis of (8.9) we conclude that transferring the body to the required position with respect to z_j in fact amounts to solving Problem 2.

The solution scheme includes the following stages.

1. Choosing a construction (8.6) of the control u_k with u_j^* of type (3.6). At this stage α_j^*, and β_j^* are undefined. As compared to the approach in [2], the u_k are simplified. They do not contain any components with y_i or, in particular, any components compensating for the gyroscopic moments of the body.

2. The preliminary choice of τ and "determining" the levels β_j^*. According to (8.8) this predetermines α_j^*, ρ_j^*. For $\tau_j = \tau$ ("equalization" of the control time for each z_j) we have

$$\alpha_j^* = \beta_j^* + 4|z_{j0}|\tau^{-2} \tag{8.10}$$

3. Estimating z_j and $\dot{z}_j$ on the set L of states of the linear system (8.5), (3.6) for all admissible $|v_j^*| \leq \beta_j^*$. At this stage we use the principle of "prescribing and subsequent confirmation" of the levels β_j^*. Only once the levels β_j^* are prescribed is it possible, in particular, to use estimates of type presented in [2] to estimate v_j^* on L. We observe that, unlike the approach in [2], it is necessary to use the principle stated above to realize the algorithm.

t	z_j^+	z_j^-	$\dot{z}_j^-$
$[0, T_j^*]$	$\lvert z_{j0}\rvert - s_j^- t^2$	$\lvert z_{j0}\rvert - s_j^+ t^2$	$2s_j^+ t$
$(T_j^*, \tau/2]$	-"-	$s_j^-(t-T_j)^2$	$2s^+T_j^*$
$(\tau/2, T_j]$	$s_j^-(t-\tau)^2$	-"-	-"-
$(T_j, \tau]$	-"-	0	-"-

4. Verification of the original restrictions (1.2) on u_k in L. For (8.6) it is natural to expect that max $|u_k|$ will be close on the set L and on the subset $L^* \subset L$ of states of the linear system (8.5), (3.6) for various combinations $v_j^* = \pm\rho_j u_j^*$ and $v_j^* \equiv 0$. It is not difficult to compute max $|u_k|$ on L^*.

As a result, we obtain an iterative algorithm for solving Problem 2.

8.5. *Construction of estimates for* v_j^*. We use the inequalities $|y_1 y_2| \leqslant 1/2\Sigma y_i^2$ (1 2 3) and the relation

$$\Sigma y_i^2 = 4\{[z_4^{-1}\Sigma(z_j\dot{z}_j)]^2 + \Sigma(\dot{z}_j)^2\} \tag{8.11}$$

which can be verified using (8.9). We obtain the following (overestimated) bounds

$$\begin{aligned} &|v_1^*| \leqslant (z_1^+ + z_4^+ r_1 + z_2^+ r_3 + z_3^+ r_2)G + \Delta_1 \\ &\Delta_1 \leqslant \tfrac{1}{2}(z_4^+\beta_1 A_1^{-1} + z_2^+\beta_3 A_3^{-1} + z_3^+\beta_2 A_2^{-1}), \quad r_1 = |A_2 - A_3| A_1^{-1} \quad (1\ 2\ 3) \\ &G = (z_4^-)^{-2}[\Sigma(z_j^+\dot{z}_j^-)]^2 + \Sigma(\dot{z}_j^-)^2, \quad z_4^{(-,+)} = [1 - \Sigma(z_j^{(+,-)})^2]^{1/2} \end{aligned} \tag{8.12}$$

The expressions for $z_j^{(+,-)}$, $\dot{z}_j^-$ (overestimated for $\dot{z}_j^-$ when $t \geqslant t_0 + T_j^*$) are listed in the table, in which

$$T_j = 2(1-\rho_j)^{-1}T_j^*, \quad T_j^* = [(2s_j^+)^{-1}|z_{j0}|(1-\rho_j)]^{1/2}, \quad s_j^{\pm} = \tfrac{1}{2}\alpha_j^*(1 \pm \rho_j)$$

The overestimated bounds (8.12) can be weakened. To this end we observe that y_i of the form (8.9) preserve the sign ($y_i \leqslant 0$ if $z_{j0} > 0$) in many cases for all admissible v_j^*. For example, if $A_2 \geqslant A_3 \geqslant A_1$ (other cases can be dealt with in a similar way), the first group of inequalities in (8.12) can be replaced by

$$|v_1^*| \leqslant [\max(z_4^+ r_1 + z_3^+ r_2, z_1^+ + z_2^+ r_3)]G + \Delta_1 \quad (1\ 2\ 3) \tag{8.13}$$

8.6. *Construction of estimates for* u_k. Calculations can be simplified if the inequality

$$E = \Sigma(u_k^2 A_k^{-2}) < \alpha = \text{const} > 0 \tag{8.14}$$

is used in place of (1.2) to estimate (8.6). Indeed, by (8.6) E is given by (8.11) if $\dot{z}_j$ is replaced by u_j^*. Thus (8.14) is satisfied if

$$E^* = [(z_{40})^{-1}\Sigma(|z_{j0}|\alpha_j^*)]^2 + \Sigma(\alpha_j^*)^2 < \tfrac{1}{4}\alpha \tag{8.15}$$

Moreover, for the integral estimate $E = E(t)$ for $t \in [t_0, t_0 + \tau]$ we can use the inequality $E \leqslant 4\{[(z_4^-)^{-1}\Sigma(z_j^+\alpha_j^*)]^2 + \Sigma(\alpha_j^*)^2\}$.

8.7. *Solvability conditions for Problem* 2. Let us summarize the above argument.

Theorem 2. If the levels α_k of the controls u_k in (8.1)–(8.3) are high enough, then for any given levels β_i of the noise v_i the rules u_k which solve Problem 2 can be constructed in the form (8.6), where u_j^* have the form (3.6) (once the subscripts are appropriately specified). This ensures precise reorientation of the body in a finite time τ for any $\mathbf{v} \in K_1$. The value of τ can be established by iteration using the algorithm presented in Section 8.4.

Corollary 2. Suppose that the levels β_i of the noise v_i in (8.2)–(8.3) are such that for the "assigned" values τ, β_j^* the estimates $|v_j^*| \leqslant \beta_j^*$ hold on the basis of inequalities of type (8.12) or (8.13). Then, given that (8.15) holds, the controls (8.6) satisfy (8.14) and ensure precise reorientation of the body in a finite time τ.

8.8. *A possible development of Problem* 2. Problem 2 assumes that the initial and final states of the body subject to rotation are states of rest. Along with this problem, in the ideal noiseless case the problem of the reorientation of a body was solved [27] for a body which fails to come to rest in the final state following rotation. In this case it is ensured only that the body "passes" through the desired angular state in three-dimensional inertial space.

This reorientation was compared [27] with the traditional one, when the body is at rest in the initial and final positions. It was shown that for identical restrictions on the controls the "non-traditional" reorientation can be realized much more quickly. Besides, the resulting control is simpler. This is of interest when designing spacecraft reorientation systems to perform operations of short duration at the instant when the body passes through the required angular position, such as photography, striking a target, data transmission, etc.

It is of theoretical and practical interest to solve such a problem in a game-theoretic setting. To do this one can use, for example, the approach proposed in [1–4] and developed in this paper (subject to a suitable modification).

9. EXAMPLE

To estimate the capabilities of the algorithm we consider the reorientation of a body with $A_1 = 4 \times 10^4, A_2 = 8 \times 10^4, A_3 = 5 \times 10^4$ (kg m^2) from the position $\mathbf{y}_0 = \mathbf{0}$, $\mathbf{z}_0 = (0.701; 0.353; 0.434; 0.432)$ to $\mathbf{y}_1 = \mathbf{0}$.

We put $\tau = 70$ (s). Using inequalities (8.13), which hold in this case, for

$$1/2\,|\,z_4 v_1 A_1^{-1} + z_2 v_3 A_3^{-1} - z_3 v_2 A_2^{-1}\,| \leq 10^{-3}\,(\mathrm{s}^{-2}) \quad (1\,2\,3) \tag{9.1}$$

we can put $\beta_j^* = 245 \times 10^{-5}$ (s^{-2}), as can be seen from computations. In this case $4E^* = 185 \times 10^{-6}$ (kg s^{-4}). It follows that the controls (8.6) satisfy (8.13) with $\alpha = 4E^*$. The "mean" value of E is equal to 123×10^{-6} (kg s^{-4}).

Using the amplifying inequalities, from (9.1) we have $\beta_1 = 39.9$; $\beta_2 = 92.9$; $\beta_3 = 57.9$ (N m). But the admissible values of v_i may be higher.

For comparison, we also performed computations following the method described in [2]. With the same restrictions (9.1) and the same τ the control levels u_k in [2] are such that inequality (8.14) is satisfied for $\alpha = 174 \times 10^{-6}$ (kg s^{-4}). Comparison shows that the proposed modification of the method in [2] is effective.

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REFERENCES

1. Vorotnikov, V. I., *Stability of Dynamical Systems with Respect to a Part of the Variables*. Nauka, Moscow, 1991.
2. Vorotnikov, V. I., Controlling the angular motion of a rigid body in the presence of noise. A game-theoretic approach. *Prikl. Mat. Mekh.*, 1994, **58**(3), 82–103.
3. Vorotnikov, V. I., The non-linear synthesis of bounded controls in the presence of noise. *Ross. Akad. Nauk Dokl.*, 1994, **337**(1), 44–47.
4. Vorotnikov, V. I., The problem of the non-linear synthesis of time-suboptimal bounded controls in the presence of noise. *Ross. Akad. Nauk. Izv. Tekhn. Kibern.*, 1994, 6, 88–116.
5. Chernous'ko, F. L., Decomposition and suboptimal control in dynamical systems. *Prikl. Mat. Mekh.*, 1990, **54**(6), 883–893.
6. Chernous'ko, F. L., Decomposition and synthesis of control in dynamical systems. *Izv. Akad. Nauk SSSR. Tekhn. Kibern.*, 1990, 6, 64–82.
7. Chernous'ko, F. L., Decomposition and suboptimal control in dynamical systems. *Ross. Akad. Nauk. Tekhn. Kibern.*, 1993, 1, 209–214.
8. Rumyantsev, V. V. and Oziraner, A. S., *Stability and Stabilization of Motion with Respect to a Part of the Variables*. Nauka, Moscow, 1987.
9. Akulenko, L. D., *Asymptotic Methods of Optimal Control*. Nauka, Moscow, 1987.
10. Vorotnikov, V. I., Problems and methods of investigating the stability and stabilization of motion with respect to a part of the variables: research trends, results and features. *Avtomatika i Telemekh.*, 1993, 3, 3–62.
11. Kovalev, A. M., Controllability of dynamical systems with respect to a part of the variables. *Prikl. Mat. Mekh.*, 1993, **57**(6), 41–50.
12. Krasovskii, N. N., *Game-theoretic Encounter Problems*. Nauka, Moscow, 1970.
13. Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., *Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1983.
14. Vorotnikov, V. I., The theory of stability with respect to a part of the variables. *Prikl. Mat. Mekh.*, 1995, **59**(4), 553–561.
15. Branets, V. N. and Shmyglevskii, I. P., *Application of Quaternions in Problems of the Orientation of a Rigid Body*. Nauka, Moscow, 1973.
16. Vorotnikov, V. I., On the stabilization of permanent revolutions of a massive rigid body with a fixed point. *Izv. Akad. Nauk SSSR. MTT*, 1985, 3, 16–18.
17. Vorotnikov, V. I., The stabilization of gyrostat orientation in a circular orbit in a Newtonian force field. *Izv. Akad. Nauk SSSR. MTT*, 1986, 3, 25–30.
18. Vorotnikov V. I., Control of the angular motion of a rigid body. *Izv. Akad. Nauk SSSR. MTT*, 1986, 6, 38–43.
19. Dwyer, T. A. W. III, Exact non-linear control of large angle rotational maneuvers. *IEEE Trans. Aut. Control*, 1984, **AC-29**(9), 769–774.

20. Kein, V. M., Parikov, A. N., and Smurovm, M. Yu., Optimal control by the method of extremum aiming. *Prikl. Mat. Mekh.*, 1980, **44**(3), 306–310.
21. Korneyev, V. A., Melikyan, A. A. and Titovskii, I. N., Stabilization of the glide path of an aircraft subject to windshear in a minimax formulation. *Izv. Akad. Nauk SSSR. Tekhn. Kibernetika*, 1985, 3, 132–139.
22. Miele, A., Wang, T. and Melvin, W. W., Optimal take-off trajectories in the presence of windshear. *J. Optim. Theory and Applications*, 1986, **49**(1), 1–45.
23. Leitmann, G. and Pandey, S., Aircraft control for flight in an uncertain environment: takeoff in windshear. *J. Optim. Theory and Applications*, 1991, **70**(1), 25–55.
24. Bulirsch, R., Montrone, F. and Persch, H. J., Abort landing in the presence of windshear as minimax optimal control problem, I. II. *J. Optim. Theory and Applications*, 1991, **70**(1), 1–23; 1991, **70**(2), 223–254.
25. Zhao, Y. and Bryson, A. E., Approach guidance in downburst. *J. Guidance, Control and Dynamics*, 1992, **15**(4), 893–900.
26. Botkin, N. D., Zarkh, M. A., Kein, V. M., Patsko, V. S. and Turova, V. L., Differential games and problems of aircraft control in the presence of windshear. *Izv. Ross. Akad. Nauk. Tekhn. Kibern.*, 1993, 1, 68–76.
27. Vorotnikov, V. I., The non-linear problem of the reorientation of an asymmetric rigid body. *Dokl. Ross Akad. Nauk*, 1997, **352**(3).

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